Problem 2.21

One of these is an impossible electrostatic field. Which one?

(a) $\mathbf{E} = k[xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}]$.

(b) $\mathbf{E} = k[y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}]$.

Here k is a (nonzero) constant with the appropriate units. For the *possible* one, find the potential, using the origin as your reference point. Check your answer by computing ∇V. [*Hint:* You must select a specific path to integrate along. It doesn't matter *what* path you choose, since the answer is path-independent, but you simply cannot integrate unless you have a definite path in mind.]

Solution

An electrostatic field must satisfy $\nabla \times \mathbf{E} = \mathbf{0}$. Calculate the curls of the given fields and see which is nonzero.

$$\begin{aligned}
\nabla \times \mathbf{E}_a &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kxy & 2kyz & 3kxz \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}(3kxz) - \frac{\partial}{\partial z}(2kyz)\right]\hat{\mathbf{x}} - \left[\frac{\partial}{\partial x}(3kxz) - \frac{\partial}{\partial z}(kxy)\right]\hat{\mathbf{y}} + \left[\frac{\partial}{\partial x}(2kyz) - \frac{\partial}{\partial y}(kxy)\right]\hat{\mathbf{z}} \\
&= [(0) - (2ky)]\,\hat{\mathbf{x}} - [(3kz) - (0)]\,\hat{\mathbf{y}} + [(0) - (kx)]\,\hat{\mathbf{z}} \\
&= -2ky\hat{\mathbf{x}} - 3kz\hat{\mathbf{y}} - kx\hat{\mathbf{z}}
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{E}_b &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ky^2 & k(2xy + z^2) & 2kyz \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}(2kyz) - k\frac{\partial}{\partial z}(2xy + z^2)\right]\hat{\mathbf{x}} - \left[\frac{\partial}{\partial x}(2kyz) - \frac{\partial}{\partial z}(ky^2)\right]\hat{\mathbf{y}} + \left[k\frac{\partial}{\partial x}(2xy + z^2) - \frac{\partial}{\partial y}(ky^2)\right]\hat{\mathbf{z}} \\
&= [(2kz) - k(2z)]\,\hat{\mathbf{x}} - [(0) - (0)]\,\hat{\mathbf{y}} + [k(2y) - (2ky)]\,\hat{\mathbf{z}} \\
&= 0\hat{\mathbf{x}} - 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
&= \mathbf{0}
\end{aligned}$$

Therefore, the electric field in part (a) is impossible. $\nabla \times \mathbf{E} = \mathbf{0}$ implies the existence of a potential function $-V$ that satisfies

$$\mathbf{E} = \nabla(-V) = -\nabla V.$$

The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). To solve for V, integrate both sides along a path between two points in space with position vectors, **a** and **b**, and use the fundamental theorem for gradients.

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l}_0 &= -\int_{\mathbf{a}}^{\mathbf{b}} \nabla V\cdot d\mathbf{l}_0 \\ &= -[V(\mathbf{b}) - V(\mathbf{a})] \\ &= V(\mathbf{a}) - V(\mathbf{b})\end{aligned}$$

In this context **a** is the position vector for the reference point (taken to be the origin $\mathcal{O}$ here), and **b** is the position vector **r** for the point we're interested in knowing the electric potential.

$$\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}_0 = V(\mathcal{O}) - V(\mathbf{r})$$

The potential at the reference point is taken to be zero: $V(\mathcal{O}) = 0$.

$$\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}_0 = -V(\mathbf{r})$$

Therefore, the potential at $\mathbf{r} = \langle x, y, z\rangle$ is

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}_0.$$

Determine the electric potential corresponding to the electric field in part (b) by integrating along some convenient path from the origin to the point of interest.

$$\begin{aligned}V_b(\mathbf{r}) &= -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}_b\cdot d\mathbf{l}_0 \\ &= -\int_{\langle 0,0,0\rangle}^{\langle x,y,z\rangle} \mathbf{E}_b\cdot d\mathbf{l}_0 \\ &= -\left(\int_{\langle 0,0,0\rangle}^{\langle x,0,0\rangle} \mathbf{E}_b\cdot d\mathbf{l}_0 + \int_{\langle x,0,0\rangle}^{\langle x,y,0\rangle} \mathbf{E}_b\cdot d\mathbf{l}_0 + \int_{\langle x,y,0\rangle}^{\langle x,y,z\rangle} \mathbf{E}_b\cdot d\mathbf{l}_0\right)\end{aligned}$$

Parameterize the three line segments.

$$\begin{aligned}&\text{Segment 1:} \qquad x_0 = s_0 \quad \text{and} \quad y_0 = 0 \quad \text{and} \quad z_0 = 0, \qquad 0 \le s_0 \le x \\ &\text{Segment 2:} \qquad x_0 = x \quad \text{and} \quad y_0 = s_0 \quad \text{and} \quad z_0 = 0, \qquad 0 \le s_0 \le y \\ &\text{Segment 3:} \qquad x_0 = x \quad \text{and} \quad y_0 = y \quad \text{and} \quad z_0 = s_0, \qquad 0 \le s_0 \le z\end{aligned}$$

Consequently,

$$\begin{aligned}V_b(\mathbf{r}) &= -\int_0^x \mathbf{E}_b(x_0,y_0,z_0)\cdot \mathbf{l}_0'(s_0)\,ds_0 - \int_0^y \mathbf{E}_b(x_0,y_0,z_0)\cdot \mathbf{l}_0'(s_0)\,ds_0 - \int_0^z \mathbf{E}_b(x_0,y_0,z_0)\cdot \mathbf{l}_0'(s_0)\,ds_0 \\ &= -\int_0^x \mathbf{E}_b(s_0,0,0)\cdot\langle 1,0,0\rangle\,ds_0 - \int_0^y \mathbf{E}_b(x,s_0,0)\cdot\langle 0,1,0\rangle\,ds_0 - \int_0^z \mathbf{E}_b(x,y,s_0)\cdot\langle 0,0,1\rangle\,ds_0.\end{aligned}$$

Plug in the electric field, evaluate the integral, and then simplify the result.

$$\begin{aligned}
V_b(\mathbf{r}) &= -\int_0^x \langle 0,0,0\rangle \cdot \langle 1,0,0\rangle\, ds_0 - \int_0^y \langle ks_0^2, 2kxs_0, 0\rangle \cdot \langle 0,1,0\rangle\, ds_0 - \int_0^z \langle ky^2, k(2xy+s_0^2), 2kys_0\rangle \cdot \langle 0,0,1\rangle\, ds_0 \\
&= -\int_0^x (0)\, ds_0 - \int_0^y (2kxs_0)\, ds_0 - \int_0^z (2kys_0)\, ds_0 \\
&= -2kx\int_0^y s_0\, ds_0 - 2ky\int_0^z s_0\, ds_0 \\
&= -2kx\left(\frac{y^2}{2}\right) - 2ky\left(\frac{z^2}{2}\right) \\
&= -kxy^2 - kyz^2
\end{aligned}$$

Therefore,

$$V_b(x,y,z) = -k(xy^2 + yz^2).$$

Check this answer by computing the partial derivatives of V_b.

$$\begin{aligned}
\frac{\partial V_b}{\partial x} &= \frac{\partial}{\partial x}(-kxy^2 - kyz^2) = -ky^2 \\
\frac{\partial V_b}{\partial y} &= \frac{\partial}{\partial y}(-kxy^2 - kyz^2) = -2kxy - kz^2 \\
\frac{\partial V_b}{\partial z} &= \frac{\partial}{\partial z}(-kxy^2 - kyz^2) = -2kyz
\end{aligned}$$

As a result,

$$\begin{aligned}
\nabla V_b &= \left\langle \frac{\partial V_b}{\partial x}, \frac{\partial V_b}{\partial y}, \frac{\partial V_b}{\partial z} \right\rangle \\
&= \langle -ky^2, -2kxy - kz^2, -2kyz\rangle \\
&= -k\langle y^2, 2xy + z^2, 2yz\rangle \\
&= -\mathbf{E}_b.
\end{aligned}$$